An Application: How to Divide a Cake

When divorcing couples split up marital property, they face a problem of fair division. Similar issues arise when nations negotiate over disputed territory, or when a gang of bank robbers divides up the loot. For concreteness, think of dividing a cake. The cake need not be uniform: part might be chocolate, part vanilla. Or some portions may have lots of raisins, others very few. So what is "fair" will depend in part upon the tastes of the parties.

If there are only two claimants, a well-known principle (attributed to King Solomon) applies. One person, chosen by toss of a coin, cuts the cake; the other has the first choice. For Abigail, the cutter, the problem is what principle to follow in making the division. As for the chooser, Bill, his principle is obvious: after Abigail has made the cut, he takes whichever piece he prefers.

If the cake is sliced in an arbitrary way, the result might be neither efficient nor equitable. For example, if one party loves raisins and the other hates them, the cake might conceivably be cut in such a way that a different division could make them both better off. And of course it might be cut in such a way that at least one claimant envies the other's slice. The question is whether Abigail, following her own self-interest, will rationally cut the cake in a way that is both efficient and envy-free.

Abigail's rational strategy will depend upon what she knows about Bill's preferences. Only the two extreme cases will be considered here: (1) Abigail has *perfect knowledge* of Bill's preferences. (2) Abigail is *completely ignorant* of Bill's preferences.

PERFECT KNOWLEDGE Suppose the cake is exactly half chocolate, half vanilla. Imagine that Abigail knows that Bill loves chocolate, whereas she herself is indifferent. Then it makes sense for her to split the cake into two portions: a smaller chocolate slice for Bill and a bigger one containing the remainder of the chocolate, plus all the vanilla, for herself. The question is, how unequal should she make the slices?

Call b the fraction of the entire cake (consisting of some but not all of the chocolate portion) she carves off for Bill. And suppose she knows that Bill's relative preference for chocolate over vanilla is always in the constant ratio r. Then Abigail will want to make Bill almost (but not quite) indifferent between his assigned chocolate portion and the remainder of the cake – the portion she intends to retain.

Bill will be indifferent if:

$$rb = \frac{1}{2} + r\left(\frac{1}{2} - b\right) \tag{16.1}$$

The left-hand side of this equation is Bill's evaluation of the chocolate portion: his fraction b of the entire cake multiplied by r, his relative preference for chocolate. The right-hand side is Bill's evaluation of the other portion. It consists of the vanilla half of the cake, entering with a weight of 1, plus the remaining chocolate fraction of the cake weighted by r.

Solving the equation, the result is:

$$b = \frac{1+r}{4r} \tag{16.2}$$

So if, for example, Abigail knows that $r = 2$ (Bill requires twice as much vanilla before being willing to give up a piece of chocolate), she should give Bill 3/8 of the cake, in the form of 3/4 of the chocolate portion. To which she would have to add a crumb, to make Bill definitely prefer his slice rather than remaining indifferent.